Semiclassical narrow-tube quantizations of time-periodic Hamiltonian systems

1994 J. Phys. A: Math. Gen. 27 7475

(http://iopscience.iop.org/0305-4470/27/22/020)

J. Phys. A: Math. Gen. 27 (1994) 7475–7490. Printed in the UK

Semiclassical narrow-tube quantizations of time-periodic Hamiltonian systems

Karl-Erik Thylwe† and Frank Bensch‡

† *Department of Mechanics, Royal Institute of Technology S-10044 Stockholm, Sweden*
‡ Instituto de Fisica 'Gleb Wataghin', UNICAMP, 13083-970 Campinas-SP, Brazil

Received 23 May 1994, in final form 19 August 1994

Abstract. An approximate semiclassical quantization procedure for time-periodic Hamiltonians is formulated, which applies to motion near stable periodic orbits of the system, corresponding to elliptic islands of the *period map*. The theory involves canonical transformations and the use of a particular *dynamical* Lewis invariant. The basic assumption in the narrow-tube approximation is the localization of the quantal wavefunction inside the elliptic islands.

1. Introduction

The dynamics of atoms or molecules in strong laser fields, as of highly excited hydrogen atoms in microwave fields, has become an important subject of study, motivated by the increasingly more sophisticated laser facilities. Various time-periodic Hamiltonian systems, with $H(t+T) = H(t)$, may serve as mathematical models for analysing the details of the dynamics involved. In the present investigation we explore a class of time-periodic oscillators with one spatial degree of freedom.

Generically time-dependent systems are known to show chaotic dynamics even for only one (spatial) degree of freedom [1]. A question of considerable fundamental interest is to what extent and in what way classical nonlinear phenomena, like chaos, reflect themselves in quantum mechanics [2]. The quantization of 'bundles' or 'cables' of vortex tubes as they occur, for example, near subharmonic resonances, i.e. when the winding frequencies of the trajectories become close to a rational multiple of the driving frequency of the system, is of particular interest to us in the present work. The original, nested vortex tubes of the forced linear oscillator are known to break up into a series of narrower ones if nonlinearities in the system are introduced. The question of how the rich variety of such subharmonic phenomena in classical mechanics is reflected in quantum mechanics does not seem to have attracted much attention. On the other hand it is quite well known that chaos is often approached through an infinite sequence of subharmonic motion (the period-doubling mechanism).

A semiclassical EBK quantization of time-periodic Hamiltonian models, using an extended, four-dimensional phase space, has recently been introduced [2–4]. A narrow-tube approximation, based on this semiclassical EBK quantization procedure, has also been presented and numerically tested by the authors [5] to understand the quasi–energy contribution from the motion near the periodic centre of an elliptic island. Our leading-order narrow-tube approximation resulted in a formula similar (but not identical) to Miller's modified periodic orbit quantum condition [6] for energy eigenvalues of *time-independent* systems. Unfortunately, Miller's derivation (periodic-orbit theory) has not yet been applied to elliptic islands of time-periodic (Hamiltonian) systems, so a rigorous comparison between

0305-4470/94/227475+16$19.50

our leading-order narrow-tube formula [5] and periodic orbit results will have to wait. Meanwhile, we demonstrate in the present work that our quasi-energy formula (equation (31) in [5]), obtained from the narrow-tube approximation of the quantization conditions given by EBK theory [2], can also be derived from the more rigorous WKB theory (see chapter 6 in [1]). Such a derivation does not involve smooth geometrical objects in a four-dimensional phase space, but requires time-dependent canonical transformations and some more details of second-order, ordinary differential equations and Floquet mappings.

Our aim is to formulate a local semiclassical quantization procedure—semiclassical narrow-tube (SCNT) quantization—applicable to elliptic islands of the relevant *period* (Poincaré) *map* of the system phase space. A typical *period map* may show several elliptic islands corresponding to classes of subharmonic and harmonic responses of the excited oscillator. All these islands may support sequences of quasi-energies, which, together with contributions from other types of regular and irregular phase-space regions, combine to a set of quasi-energies for the whole system. We believe that such a semiclassical narrow-tube quantization can provide some impact on further development of the periodic-orbit theory for time-periodic Hamiltonian systems [1, 3].

In section 2 the basic narrow-tube Hamiltonian is derived for a general system of one spatial degree of freedom. The resulting Hamiltonian represents a parametrically excited harmonic oscillator—the classical Hill oscillator. Details are worked out explicitly for the excited Duffing oscillator. The Lewis invariant, which is an integral of motion for the Hill oscillator problem, is presented in section 3. Section 4 introduces a final transformation of the basic narrow-tube Hamiltonian to its canonical form, which is then quantized in section 5. The resulting formula for the quasi-energies supported by elliptic islands of the phase-space period map is numerically tested in section 6, and the result is compared to those of semiclassical EBK theory and quantum theory. The applicability and certain limitations of the narrow-tube theory are discussed in the conclusions.

In two appendices we draw attention to an amplitude–phase decomposition of the local motion about the periodic orbit centre of an elliptic island. Appendix 1 deals with the basic equations governing the behaviours of amplitude and phase functions which are relevant for the analysis of the Lewis invariant. Appendix 2 makes use of the amplitude–phase decomposition for analysing winding behaviour of tube-like motion appearing in the local period map of phase space. In this way a quantity like the classical angular winding frequency can be identified in the narrow-tube formula for the quantized quasi-energies.

2. Basic narrow-tube Hamiltonian

This section describes the initial canonical transformations to a non-inertial system of coordinates with origin locked to a periodic motion of an elliptic centre. The period T_p of the centre motion may in principle be any rational multiple of the 'external' excitation period T. The theory is outlined for one spatial degree of freedom and explicit results for the excited Duffing oscillator are given.

2.1. General formulation

We consider a time-periodic Hamiltonian system of a single spatial degree of freedom

$$H(p, q, t) = H(p, q, t + T) \tag{2.1}$$

where $T = 2\pi/\omega$ is the period of some unspecified excitation (forcing). The system may be strongly nonlinear in the canonical coordinate q, but the Hamiltonian is assumed to be

quadratic in the canonical momentum p. Hence, we may introduce a time-periodic potential $V(q,t)$ according to the decomposition:

$$H(p,q,t) \equiv \tfrac{1}{2}p^2 + V(q,t). \tag{2.2}$$

The Hamiltonian equations of classical motion take the general form:

$$\dot{q} = p \qquad \dot{p} = -\frac{\partial V}{\partial q}. \tag{2.3}$$

It is well known that excited Hamiltonian oscillators possess several periodic (harmonic, subharmonic) responses, some of which are stable and define so-called elliptic islands of the *period map* (mapping at a period [7]) of phase space. We now consider the immediate neighbourhood of an arbitrary periodic response (the centre of an elliptic island), and introduce a relative canonical coordinate λQ and momentum λP through the transformation:

$$q = q_p(t) + \lambda Q \qquad p = \dot{q}_p(t) + \lambda P \tag{2.4}$$

where $q_p(t) = q_p(t + T_p)$ is the periodic centre motion and λ is a book-keeping parameter. Several generating functions accomplishing a proper canonical transformation are possible to construct, with equivalent end results, but the most elegant one found by the authors [5] is

$$F_2(\lambda P, q, t) \equiv (q - q_p(t))(\dot{q}_p(t) + \lambda P) \tag{2.5}$$

which satisfies the equations

$$p = \frac{\partial F_2}{\partial q} \qquad \lambda Q = \frac{\partial F_2}{\partial(\lambda P)}. \tag{2.6}$$

The transformed Hamiltonian $K(\lambda P, \lambda Q, t)$ is given according to the general formula:

$$K(\lambda P, \lambda Q, t) = H(p,q,t) + \frac{\partial F_2}{\partial t} \tag{2.7}$$

where, with proper substitutions, we find

$$\frac{\partial F_2}{\partial t} = -\lambda P \dot{q}_p(t) + \lambda Q \ddot{q}_p(t) - \dot{q}_p^2(t). \tag{2.8}$$

So far the Hamiltonian is obtained by exact transformations. The narrow-tube Hamiltonian pertaining to the stable periodic centre motion $q_p(t)$ is introduced next. We expand the original Hamiltonian up to quadratic terms in λ, yielding

$$H(p,q,t) \approx H(\dot{q}_p(t), q_p(t), t) + \lambda[\dot{q}_p(t)P - \ddot{q}_p(t)Q] + \tfrac{1}{2}\lambda^2\left[P^2 + \left(\frac{\partial^2 V}{\partial q^2}\right)_{\lambda=0} Q^2\right]. \tag{2.9}$$

Combining (2.7), (2.8) and (2.9), we obtain the narrow-tube limit of the transformed Hamiltonian, i.e.

$$\begin{aligned} K(\lambda P, \lambda Q, t) &\approx K_0(\lambda P, \lambda Q, t) \\ &= \frac{\lambda^2}{2}\left[P^2 + \left(\frac{\partial^2 V}{\partial q^2}\right)_{\lambda=0} Q^2\right] - L_p \end{aligned} \tag{2.10}$$

with the Lagrangian of the centre motion given by

$$L_p = \dot{q}_p^2(t) - H(\dot{q}_p(t), q_p(t), t). \tag{2.11}$$

We notice that the relevant period of the transformed Hamiltonian is T_p rather than T. The Lagrangian L_p appears as an explicit function of time alone and is totally irrelevant for the local equations of motion about the periodic orbit. In fact, trajectories in the neighbourhood of the centre motion satisfy a Hill-type differential equation:

$$\ddot{Q} + \left(\frac{\partial^2 V}{\partial q^2}\right)_{\lambda=0} Q = 0. \tag{2.12}$$

The time-periodic coefficient of this linear differential equation generally has the same period T_p as the periodic centre motion.

2.2. *The forced Duffing oscillator*

Here we consider a time-periodic Duffing oscillator described by the Hamiltonian

$$H_D(p, q, t) = \tfrac{1}{2}(p^2 + kq^2 + \tfrac{1}{2}sq^4) - rq\cos(t) \tag{2.13}$$

which was considered in this context by the present authors in a recent paper [5].

The Duffing oscillator contains the linear force constant k, the excitation amplitude r and the strength of the nonlinearity s. The correct Hamiltonian corresponding to the relative motion is obtained from the time-dependent canonical transformation (2.5), and is given by:

$$K(\lambda P, \lambda Q, t) = \frac{\lambda^2}{2}(P^2 + (k + 3sq_p^2)Q^2) + \lambda^3 sq_p Q^3 + \frac{\lambda^4}{4}sQ^4 - L_p. \tag{2.14}$$

The centre motion Lagrangian, L_p, depends only on the underlying periodic trajectory,

$$L_p(t) = \tfrac{1}{2}\dot{q}_p^2 - \tfrac{1}{2}kq_p^2 - \tfrac{1}{4}sq_p^4 + q_p r\cos(t) \tag{2.15}$$

and does not enter into the dynamical equations of motion. The resulting narrow-tube Hamiltonian for the Duffing oscillator (with $\lambda = 1$) is

$$K_0(P, Q, t) = \tfrac{1}{2}p^2 + \tfrac{1}{2}(k + 3sq_p^2)Q^2 - L_p. \tag{2.16}$$

The Hamiltonian equations of motion result in the Hill-type differential equation:

$$\ddot{Q} + (k + 3sq_p^2)Q = 0. \tag{2.17}$$

Equation (2.17) describes the trajectories close to the periodic response $q_p(t)$ and is identical to the linear variational equation of standard stability theory.

In order to explicitly quantize the narrow-tube Hamiltonian we invoke the so-called Lewis invariant [3, 8, 9]. This quantity is closely connected to an amplitude–phase decomposition of the solutions of the Hill-type equation (2.12), as presented in some detail in section 3.

3. Dynamical Lewis invariant

The Lewis invariant [8] is a constant of motion, which is expressed in terms of any solution $Q(t)$ to the Hill equation (2.12) and a *class* of amplitude functions $\rho(t)$ as explained in this section (see also appendix 1). In our presentation we like to point out the less known fact that the classical Lewis invariant contains a free parameter (angular momentum parameter Λ) independent of our specification of the functions $Q(t)$ and $\rho(t)$ by initial conditions. Unfortunately, this arbitrariness conflicts with the idea that each closed curve of the period map belongs to a well defined value of the constant of motion (and of the quasi-energy). The problem, as we shall see, is solved here by requiring that the invariant set in $(Q, \dot{Q})$-phase space of initial conditions, corresponding to a given value of the Lewis invariant, should be identical to the set of points produced by the (iterated) period map when applied to any point in the initial set. In this way, we single out a *dynamical* Lewis invariant with a specified parameter value $\Lambda = \Lambda_p$. Note, however, that we have to make sure in the next section that the Lewis invariant fits into the canonical scheme, before the (WKB) quantization technique can be applied. We therefore discuss briefly the canonical Lewis invariant and its relation to area-preserving phase-space flows.

To begin with, Hill's equation (2.12) is written conveniently as

$$\ddot{Q} + \omega_0^2(t)Q = 0 \tag{3.1}$$

with an assumed reflection-symmetric coefficient

$$\omega_0^2(t) = \left(\frac{\partial^2 V}{\partial q^2}\right)_{\lambda=0} \qquad \omega_0^2(t) = \omega_0^2(-t). \tag{3.2}$$

The amplitude function $\rho(t)$, defining the fundamental solution matrix corresponding to (3.1) and, hence, any solution $Q(t)$, satisfies the Milne equation (see appendices 1 and 2).

$$\ddot{\rho} + \omega_0^2(t)\rho - \frac{\Lambda^2}{\rho^3} = 0 \tag{3.3}$$

with initial conditions $\rho(0) = 1$, $\dot{\rho}(0) = 0$.

We now study trajectories described by (3.1) and which correspond to an unspecified class of initial conditions

$$Q(0) = Q_0(\alpha) \qquad \text{and} \qquad \dot{Q}(0) = P_0(\alpha) \tag{3.4}$$

later to be identified with a given constant of motion. The differential equations (3.1) and (3.3) differ by a single nonlinear term which renders their solutions $Q(t)$ and $\rho(t)$, respectively, quite different behaviours in general. If, on the other hand, Λ assumes the particular value zero, the Wronskian $W(Q, \rho) = \dot{Q}\rho - Q\dot{\rho}$ actually becomes a constant of motion. For general Λ this is not true. By multiplying the equations (3.1) and (3.3) by ρ and Q, respectively, followed by a subtraction, the result is

$$\dot{W}(Q, \rho) = -\Lambda^2 \frac{Q}{\rho^3} \tag{3.5}$$

which shows that the Wronskian $W(Q, \rho)$ is not a constant for $\Lambda \neq 0$. Accidentally, equation (3.5) can be immediately integrated with respect to time if it is multiplied by twice the Wronskian itself. For the right-hand member of (3.5) we make use of the identity

$$2W(Q, \rho)\frac{Q}{\rho^3} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Q^2}{\rho^2}\right) \tag{3.6}$$

and arrive at the first integral

$$W^2(Q,\rho) = -\Lambda^2\left(\frac{Q^2}{\rho^2}\right) + \text{const}. \tag{3.7}$$

In other words, we have derived the Lewis invariant I:

$$I = \frac{1}{2}\left[W^2(Q,\rho) + \Lambda^2\left(\frac{Q^2}{\rho^2}\right)\right] \tag{3.8}$$

where an extra factor of one half has been introduced by convenience. The value of the invariant depends on the angular momentum parameter Λ as well as the initial conditions for both $Q(t)$ and $\rho(t)$. To show this we write a solution $Q(t)$, satisfying initial conditions (3.4), in terms of the two independent principal solutions $Q^{(1)}(t)$ and $Q^{(2)}(t)$ of appendix 2:

$$Q(t) = Q_0(\alpha)Q^{(1)}(t) + P_0(\alpha)Q^{(2)}(t). \tag{3.9}$$

Consequently, we have

$$I = \tfrac{1}{2}(P_0^2(\alpha) + \Lambda^2 Q_0^2(\alpha)). \tag{3.10}$$

We would like a *dynamical* Lewis invariant which is uniquely defined by initial conditions of the trajectory. To eliminate the arbitrary Λ dependence we consider in more detail the invariant sets of points in phase space corresponding to different values of Λ in the Lewis invariant. Obviously, there exists a closed curve of initial conditions corresponding to a given value of the invariant I. This initial cross-section curve is given, for example, by the parametrization:

$$Q_0(\alpha) = \Lambda^{-1}\sqrt{2I}\cos(\alpha) \qquad P_0(\alpha) = -\sqrt{2I}\sin(\alpha). \tag{3.11}$$

In order to see the Λ dependence more clearly, it may be advantageous to express the set of initial points (3.11) in terms of one member $(Q_0(0), P_0(0)) = (Q_0(0), 0)$, say. In such a representation we have:

$$Q_0(\alpha) = Q_0(0)\cos(\alpha) \qquad P_0(\alpha) = -\Lambda Q_0(0)\sin(\alpha). \tag{3.12}$$

From an analysis of the phase-space *period map* in appendix 2 we find that the initial cross-section curve (3.12) becomes identical to the image set of repeated period mappings $M \circ M \circ \cdots$ for the member $(Q_0(0), 0)$ only if $\Lambda = \Lambda_p$, so that the amplitude function ρ is T_p-periodic. This means that the fixed-time cross sections of the vortex tube associated to the dynamical Lewis invariant are T_p-periodic.

How can we now interpret the Lewis invariant? One aspect of Hamiltonian flows that we should consider is Liouville's theorem on preservation of phase-space area [1]. The area enclosed by the phase-space curve corresponding to a given value of the invariant I in (3.10), at any fixed time t, is directly obtainable from (3.9) and (3.11) by integrating over the possible values of the initial angle α. We find

$$\int_0^{2\pi} \dot{Q}(\alpha)\frac{\mathrm{d}Q(\alpha)}{\mathrm{d}\alpha}\,\mathrm{d}\alpha = 2\pi I\Lambda^{-1}\det Q(t) \tag{3.13}$$

at any time. It is shown in appendix 2 that the fundamental solution matrix satisfies $\det Q(t) = 1$, so the enclosed phase-space area is actually time independent, as it should. The relation (3.13) also suggests that the invariant I in (3.10) is *not* the *canonical* action integral of motion associated to the cross-section area of the tube, rather the combination $I\Lambda^{-1}$ is. The canonical Lewis invariant, which can be singled out from considerations of Hamiltonian period maps, will appear as the 'optimal' canonical formulation in section 4.

In closing this section, we discuss how the parameter Λ_p has to be calculated. We first, as an illustration, consider the time-independent example where $\omega_0^2(t) = k$ in (3.1) and (3.3). Then there exists a real positive constant solution $\rho = (\Lambda/\sqrt{k})^{1/2}$ of the Milne equation for any parameter value Λ. For this constant solution to satisfy the initial condition $\rho(0) = 1$, we require a particular value of Λ, namely

$$\Lambda_p = \sqrt{k}. \tag{3.14}$$

The negative solution can be discarded without loss of generality. Only with this choice of Λ_p do we recover the trivial amplitude–phase representation of the simple harmonic oscillator solution. The value of the dynamical invariant I for this case is therefore the harmonic oscillator energy:

$$I = \tfrac{1}{2}(P_0^2(\alpha) + kQ_0^2(\alpha)) \tag{3.15}$$

while the canonical Lewis invariant, discussed earlier in this section, is the oscillator action integral $I/\sqrt{k}$.

In the general problem, where $\omega_0^2(t)$ is time periodic, numerical calculations become important. One may still define a straightforward procedure for determining Λ_p based on the usual Newton–Raphson iteration for locating fixed points in the proper $(\rho, \dot{\rho})$ Poincaré surface of section. This is accomplished simply by starting with a given value $\Lambda = 1$ in the integration of the Milne equation for determining the periodic solution ρ_1 by standard numerical routines. Then there is a multiplicative scaling relation between the solution ρ_1 and the proper one ρ_p. One may verify from the Milne equation (3.3) that the relation is $\rho_p = \sqrt{\Lambda_p}\rho_1$. Hence, from the initial condition on ρ_p given by (A.10) in appendix 2, and the numerically located fixed point $\rho_1(0)$ in the Poincaré surface of section, we find

$$\Lambda_p = \rho_1^{-2}(0). \tag{3.16}$$

4. Canonical narrow-tube oscillator

Here, and in section 5, we can work out consequences of the fact that the classical, and quantal, Hill oscillators are non-trivially separable models. Let us consider the basic narrow-tube Hamiltonian $K_0(P, Q, t)$ in (2.10) (with $\lambda = 1$) corresponding to a parametrically excited, classical oscillator written as

$$K_0(P, Q, t) = \tfrac{1}{2}[p^2 + \omega_0^2(t)Q^2] - L_p. \tag{4.1}$$

The Hamilton equations of motion result in the Newtonian equation of motion (2.12) with initial conditions required on $P(= \dot{Q})$ and Q. It is desirable to transform the Hamiltonian in such a way as to make optimal use of the Lewis invariant. A canonical transformation to action-angle coordinates is possible to do, but we will obtain the same final expression

for the Hamiltonian by another conjugate pair of Cartesian-type coordinates (R, S), defined by

$$R = W/\sqrt{\Lambda_p} = (P\rho(t) - Q\dot{\rho}(t))/\sqrt{\Lambda_p} \tag{4.2}$$

and

$$S = \sqrt{\Lambda_p}Q/\rho(t) \tag{4.3}$$

with ρ being the periodic Milne solution corresponding to $\rho(0) = 1$, $\dot{\rho} = 0$ and $\Lambda = \Lambda_p$ in (3.3), as before. In terms of these new coordinates, the dynamical Lewis invariant takes the form

$$I = \Lambda_p \Im \qquad \text{where } \Im = \tfrac{1}{2}[R^2 + S^2]. \tag{4.4}$$

Later we confirm that $\Im$ is the canonical Lewis invariant. A time-dependent generating function $G_2(R, Q, t)$ for the canonical transformation (4.2) and (4.3) is provided by

$$G_2(R, Q, t) = \frac{\sqrt{\Lambda_p}}{\rho(t)}RQ + \frac{\dot{\rho}(t)}{2\rho(t)}Q^2. \tag{4.5}$$

We may also confirm that $G_2(R, Q, t)$ satisfies

$$\frac{\partial G_2}{\partial Q} = \frac{1}{\rho(t)}W + \frac{\dot{\rho}(t)}{\rho(t)}Q \equiv P \tag{4.6}$$

and

$$\frac{\partial G_2}{\partial R} = \frac{\sqrt{\Lambda_p}}{\rho(t)}Q \equiv S. \tag{4.7}$$

After some straightforward analysis, the transformation of the Hamiltonian now gives

$$\begin{aligned} K_{NT}(R, S, t) &= K_0(P, Q, t) + \frac{\partial G_2}{\partial t} \\ &= \frac{\Lambda_p}{2\rho^2(t)}[R^2 + S^2] - L_p \end{aligned} \tag{4.8}$$

and from (4.4) we see that the transformed Hamiltonian can also be written as

$$K_{NT}(\Im, t) = \Lambda_p \Im/\rho^2(t) - L_p \tag{4.9}$$

The latter is the optimal form of the narrow-tube Hamiltonian. Note that the correct canonical angle corresponding to the action variable $\Im$ is the phase function φ in the amplitude–phase decomposition of appendix 1, since Hamilton's equations of motion now give

$$\dot{\varphi} = \frac{\Lambda_p}{\rho^2(t)} \tag{4.10}$$

in agreement with formula (A.4).

5. Semi-classical quantization

Let us consider the quantum oscillator corresponding to the classical narrow-tube oscillator of the preceding section. Instead of using the original operator $\hat{P}$, $\hat{Q}$ and $\hat{K}_0$, we start from the transformed classical Hamiltonian function $K_{NT}(R, S, t)$ in (4.8) to find the operator equation:

$$\left\{\frac{\Lambda_p}{2\rho^2(t)}[\hat{R}^2 + \hat{S}^2] - L_p\right\}\psi(S,t) = \hat{K}_{NT}(t)\psi(S,t) \tag{5.1}$$

with boundary conditions specified by

$$\psi(S,t) \to 0 \qquad \text{as } |S| \to \infty \tag{5.2}$$

and $\hat{R}$ and $\hat{S}$ and $\hat{K}_{NT}$ being the new quantum operators. The Schrödinger equation can be constructed from (5.1) in the usual way with the substitutions:

$$\hat{S} \to S, \hat{R} \to -i\hbar\frac{\partial}{\partial S} \qquad \text{and} \qquad \hat{K}_{NT} \to i\hbar\frac{\partial}{\partial t}. \tag{5.3}$$

In this problem, the eigenfunction $\psi(S, t)$ satisfies the additional equation

$$\tfrac{1}{2}[\hat{R}^2 + \hat{S}^2]\psi(S,t) = \Im\psi(S,t) \tag{5.4}$$

where $\Im$ is an eigenvalue of the Lewis-invariant operator $\hat{\Im} \equiv \hat{I}/\Lambda_p$. The latter eigenvalue problem is solved in the first step. The differential equation resulting from (5.3) and (5.4) is given by:

$$\frac{\partial^2}{\partial S^2}\psi(S,t) + \frac{1}{\hbar^2}[2\Im - S^2]\psi(S,t) = 0 \tag{5.5}$$

which can be quantized using WKB methods for ordinary differential equations [1]. The semi-classical condition, which follows from the requirement that the wavefunction is bounded, can be expressed as

$$\frac{1}{\hbar}\int_{-\sqrt{2I}}^{\sqrt{2I}}(2\Im - S^2)^{1/2}\,dS = (n + \tfrac{1}{2})\pi. \tag{5.6}$$

The well known result from the WKB quantization condition (5.6) is

$$\Im = (n + \tfrac{1}{2})\hbar \qquad n = 0, 1, 2, \ldots. \tag{5.7}$$

If we had started from action-angle coordinates we would have some problem to understand the boundary conditions (5.2) in terms of them. Of course, when the global oscillator problem is considered, the quantization condition (5.6) must be seen as an approximate one. There is an analogy to the problem of quantizing a time-dependent multi-well Hamiltonian system. A quantization in a single (local) well may be appropriate if surrounding barriers are sufficiently large. With this picture in mind we believe that the narrow-tube model is appropriate near the centres of well isolated elliptic islands of the Poincaré cross section.

From (5.1), (5.3), (5.4) and (5.7) we obtain the simplified partial differential equation:

$$\left\{ \frac{\Lambda_p(n+\frac{1}{2})\hbar}{\rho^2(t)} - L_p \right\} \psi(S,t) = i\hbar \frac{\partial}{\partial t}\psi(S,t) \tag{5.8}$$

which, without further approximation, gives a local narrow-tube wavefunction of the form

$$\psi(S,t) = U(S) \exp\left(-i(n+\tfrac{1}{2}) \int_0^t \frac{\Lambda_p}{\rho^2(t')}\, dt' \right) \exp\left[\frac{i}{\hbar} \int_0^t L_p(t')\, dt' \right] \tag{5.9}$$

where $U(S)$ represents a time-independent harmonic oscillator (WKB or exact quantum mechanical) wavefunction. We now turn to quantization in time and the notion of quasi-energies.

So far we have not fully exploited the fact that the narrow-tube Hamiltonian (4.1) has a periodic coefficient $\omega^2(t) = \omega^2(t + T_p)$ and that we assume an underlying real, periodic Milne solution with the same period T_p. When this is so, the phase of (5.9) is subject to an equal rate of change in any time interval of length T_p. By the introduction of the classical winding frequency W (time-averaged angular frequency) in (A.22), which can also be written

$$W = \frac{1}{T_p} \int_t^{t+T_p} \frac{\Lambda_p}{\rho^2(t')}\, dt' \tag{5.10}$$

with arbitrary t, we may further simplify the time dependence of the exponential factors in (5.9). Hence, introducing a corresponding constant energy ε_n, conjugate to the time variable, the phase increment after one period T_p takes the form:

$$\frac{\varepsilon_n T_p}{\hbar} = (n+\tfrac{1}{2}) W T_p - \frac{1}{\hbar} \int_t^{t+T_p} L_p(t')\, dt'. \tag{5.11}$$

Hence, from (5.10) and (5.11), the so-called principal quasi-energy ε_n is given by the formula

$$\varepsilon_n = \hbar W (n+\tfrac{1}{2}) - \frac{1}{T_p} \int_0^{T_p} L_p(t')\, dt'. \tag{5.12}$$

The Floquet decomposition of the wavefunction [4, 5] is a trivial matter in this case. It may be convenient to add and subtract a contribution $\varepsilon_n t/\hbar$ in the time phase in (5.9), so that a simple phase factor in the wavefunction can be extracted out, leaving the remaining factor T_p-periodic in time. Since the quantal phase increment of the remaining periodic factor is defined only mod 2π, there is in fact a whole family of equivalent quantal quasi-energies according to the formula

$$\varepsilon_{n,m} = \hbar W (n+\tfrac{1}{2}) - \frac{1}{T_p} \int_0^{T_p} L_p(t')\, dt' + \frac{2\pi}{T_p} m\hbar \tag{5.12}$$

with $m = \pm 0, \pm 1, \ldots$ and $n = 0, 1, 2, \ldots$.

Consequently, the quantal quasi-energy is the time-averaged rate of phase increase (mod 2π) multiplied by $\hbar$. Classically, the principal quasi-energy is also interpreted as the time-averaged Hamiltonian $K_{NT}(\mathfrak{S}, t)$, with the quantized value of the canonical Lewis invariant (5.7). We expect the quasi-energy formula (5.12) to be valid for harmonic as well as subharmonic elliptic islands of the period map, since the trajectory period, rather than the forcing period, enters naturally in the derivation of (5.12). One would like to expect this result to also be derivable from periodic-orbit theory, taking into account the modifications suggested by Miller [8].

6. Numerical example

In a numerical illustration of the narrow-tube quantization formula (5.12) we consider a particular Duffing oscillator (equation (2.13)) with parameters $k = 0$, $s = 1$, and $r = 0.5$. This system has previously been studied semiclassically by Bensch *et al* [4] using the EBK quantization method. In their table 1 [4] both semiclassical ε_{EBK} and quantal ε_{QM} quasi-energy results are compared in the form of quasi-angles $\theta \equiv \varepsilon T_{\text{p}}/\hbar (\text{mod } 2\pi)$ with $\hbar = 0.015$.

The Poincaré surface of section for the system is shown in figure 1. Two almost regular regions, separated by a chaotic one, can be identified. The stable, harmonic island contains 24 quasi-regular states [4], and the large chaotic region corresponding to some 80 quantal states. Outside of the chaotic region there are an infinity of tube-like states which are without reach of the narrow-tube approximation. We focus our discussion on the narrow-tube quantization of the island corresponding to a stable periodic orbit centre at $q_{\text{p}}(0) = 1.402\,064\,76$, with period $T_{\text{p}} = T$ equal to that of the forcing. In table 1 we report the principal ($m = 0$) quasi-energy results $\varepsilon_{\text{NT}} (= \varepsilon_{n,0})$ and compare them to the corresponding EBK and quantum mechanical quasi-energies converted from table 1 in [4]. The periodic Milne solution used in the narrow-tube formula is obtained with $\Lambda_{\text{p}} = 6.671\,405\,9$.

The first few quasi-energy states are excellently described by the narrow-tube formula. For larger quantum numbers, as the tubes get wider, the agreement gradually becomes worse. The realistic sizes of the tube cross-sections in the elliptic island are illustrated in

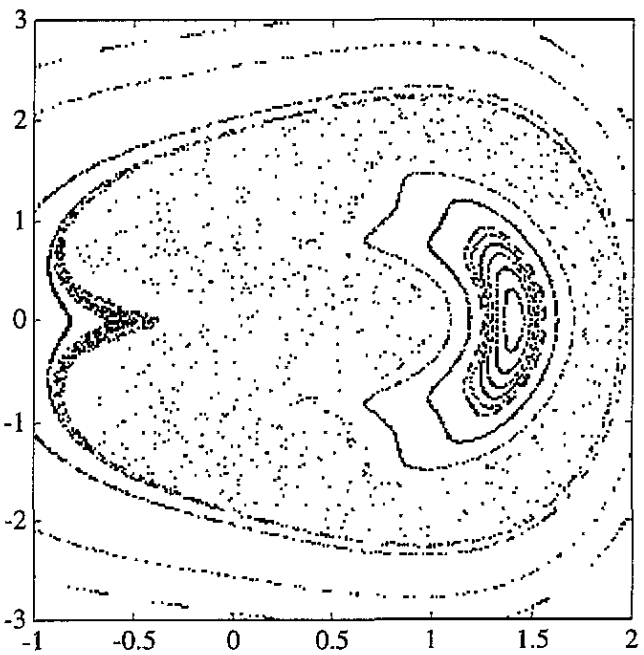

Figure 1. Phase-space period map of the excited Duffing oscillator (equation (2.13)) with system parameters $k = 0$, $s = 1$, and $r = 0.5$. Quantized quasi-energies supported by the interior quasi-regular region are studied numerically by narrow-tube formulae, and results are compared in table 1. Quantized tori corresponding to $n = 0$–4, 10 and 20, centred on the $1T$-harmonic response, are illustrated in the figure.

figure 1 for quantum numbers $n = 0$–4, 10, 20. While the EBK quantization method actually uses a representative trajectory of each quantized tube, the narrow-tube quantization method does not. It needs merely the periodic-orbit centre and a single periodic Milne solution for *all* quasi-energy states contained in the elliptic island.

Table 1. Some results of a narrow-tube quantization of the quasi-energy for the system illustrated in figure 1. For selected values of the quantum number n and $m = 0$ in (5.12), these results ε_{NT} are compared to semiclassical EBK (ε_{EBK}) and quantal results (ε_{QM}) originally calculated in [4].

n	ε_{NT}	ε_{EBK}	ε_{QM}
0	−0.457 50	−0.457 497	−0.457 497
1	−0.436 55	−0.436 559	−0.436 558
2	−0.415 60	−0.415 635	−0.415 630
3	−0.394 65	−0.394 714	−0.394 713
4	−0.373 70	−0.373 805	−0.373 806
5	−0.352 75	−0.352 912	−0.352 910
—	—	—	—
10	−0.248 00	−0.248 593	−0.248 591
11	−0.227 06	−0.227 761	−0.227 761
12	−0.206 11	−0.206 945	−0.206 944

7. Conclusions

Sufficiently close to any stable periodic orbit the time-periodic Hamiltonian is separable in a rigorous sense. In a local narrow-tube quantization one has to impose boundary conditions on the wavefunction outside the elliptic island considered. The only obvious boundary condition is that the wavefunction vanishes in the exterior region, as is typical for low-lying energy levels of a local potential well. Hence, the approximate narrow-tube formula for the quasi-energy should be more accurate when applied to larger elliptic island, and when the unit quantum area of phase space becomes smaller. This was illustrated in our numerical example. If the conditions mentioned above are satisfied, the narrow-tube result still holds when, accidentally, the winding frequency becomes a rational multiple of $2\pi/T_p$ so that a continuous family of periodic orbits appears near the periodic centre motion.

The boundary conditions used for the wavefunction in the narrow-tube quantization are no longer relevant if the centre motion is too close to bifurcation and/or if the vortex tube approaches the separatrix manifold. Unstable (hyperbolic) periodic orbits and the corresponding separatrices act as barriers, separating elliptic islands from each other.

We would also like to emphasize that elliptic islands corresponding to harmonic and subharmonic responses are treated analogously in the semiclassical narrow-tube quantization procedure. The cable of vortex tubes appearing as a periodic geometrical object of period T can be seen as a single periodic tube of period T_p.

Acknowledgments

Support by the Swedish Natural Science Research Council is gratefully acknowledged. One of us (F Bensch) would also like to thank Alexander von Humbolt Foundation and FAPESP for financial support. Finally, we appreciate the suggestions by the referees to improve the presentation of our results.

Appendix 1. Amplitude–phase decomposition and the Milne equation

In this appendix we introduce an amplitude–phase decomposition of the solutions describing the trajectories winding along the vortex tubes. It is well known for time-independent harmonic oscillator systems that the oscillatory motion can be expressed in terms of action-angle variables, so that the canonical angle appears as the phase of the oscillation and the action variable determines the (constant) amplitude. Our goal in appendices 1 and 2 for the parametrically driven harmonic oscillator is to find an analytic description of the phase development of the solution which can be defined as a canonical angle variable for the winding motion on the narrow tube. In appendix 1 we first introduce the general amplitude–phase substitution, where it turns out that the phase development of a solution is coupled to the amplitude function in a characteristic way.

The differential equation describing the trajectories is given by equation (3.1) together with equation (3.2), which for later use explicitly assume a time-reflection symmetry of the periodic coefficient $\omega_0^2(t)$. Sometimes this symmetry can be achieved by a shift of the time origin, but the detailed understanding of what restrictions this symmetry requirement imposes on the periodic orbit is beyond the scope of the present paper. With an amplitude–phase decomposition of any solution Q we write

$$Q = \rho\cos(\varphi + \alpha) \qquad \text{with } \varphi(0) = 0 \tag{A.1}$$

and α is an unspecified initial phase. One can understand that the ansatz (A.1) makes sense if $\omega_0^2(t) > 0$ for all times, since the solutions are then expected to oscillate quasi-periodically about the origin. The original differential equation (3.1) can now be replaced by an equivalent one obtained by substituting (A.1) into (3.1)

$$(\ddot{\rho} - \rho\dot{\varphi}^2 + \omega_0^2(t)\rho)\cos(\varphi + \alpha) - (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\sin(\varphi + \alpha) = 0. \tag{A.2}$$

Since we in one equation have replaced a single quantity $Q(t)$ by two new ones, $\rho(t)$ and $\varphi(t)$, we have the freedom to impose an additional, arbitrary condition that can help us solve (A.2). The obvious choice of a subsidiary condition in the phase–amplitude approach is the relation:

$$\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi} = 0 \tag{A.3}$$

which is equivalent to the relation

$$\dot{\varphi} = \frac{\Lambda}{\rho^2} \qquad \Lambda = \text{const}. \tag{A.4}$$

This is the simple relation between the angular frequency and the amplitude function for linear differential equations. Our freedom still allows us to specify the value of the angular momentum constant Λ later on. As a result of these manipulations, the exact equation (A.2) is replaced by the *Milne equation*:

$$\ddot{\rho} + \omega_0^2(t)\rho - \frac{\Lambda^2}{\rho^3} = 0 \tag{A.5}$$

for the amplitude function $\rho(t)$. This is a nonlinear equation similar to the radial part of Newton's law in cylindrical coordinates, and the solutions may behave quite differently

depending on the initial conditions. Of course, we would like the amplitude function to be well behaved and as slowly varying as possible. In many applications we would suggest an approximate, adiabatic solution of the Milne equation, i.e.

$$\rho_a(t) = \sqrt{\frac{\Lambda}{\omega_0(t)}} > 0 \tag{A.6}$$

which is periodic for all values of the 'angular momentum' parameter Λ. Note here that if $\omega_0^2(t) < 0$ for all times, the positive adiabatic amplitude function corresponds to an imaginary value of the parameter Λ.

The approximation (A.6) may not be sufficiently accurate when occasionally $\omega_0^2(t)$ becomes small. On the other hand, exact solutions may be very complicated functions of time in general. If there exist exact, periodic functions, like the approximate $\rho_a(t)$, one would be tempted to use them. At this stage in exploring the amplitude–phase decomposition we settle with the existence of a symmetric solution $\rho(-t) = \rho(t)$ (and $\dot\rho(-t) = -\dot\rho(t)$), which is directly established by the time-reflection symmetry of the differential equation (recall equation (3.2)). As a consequence, a phase function $\varphi(t)$ can be defined from (A.4) which is subject to the odd symmetry:

$$\varphi(t) = -\varphi(-t) = \Lambda \int_0^t \rho^{-2}(t')\,dt'. \tag{A.7}$$

We note that the angular momentum parameter Λ is still arbitrary. We need Floquet theory and the phase-space *period map* to find a formal condition on Λ. This will be dealt with in appendix 2.

Appendix 2. The (narrow-tube) period T_p map

In appendix 2 we use the exact amplitude–phase decomposition to derive a representation of the phase-space period map. Based on results from Floquet theory [7], the amplitude–phase representation is simplified by using a periodic amplitude function $\rho(t)$. The resulting phase function turns out to describe the correct winding motion on the tube.

From the theory of linear differential equations, any solution of (3.1) (or (2.12)) can be described by only two independent ones and their derivatives. This fact makes it possible to construct a unique stroboscopic map of phase-space positions at times $t = nT_p$, where T_p is the period of the underlying orbit $q_p(t)$, and n is any (positive and negative) integer.

We take as independent solutions $Q^{(1)}(t)$ and $Q^{(2)}(t)$, satisfying

$$\begin{aligned} Q^{(1)}(0) &= 1 \qquad \text{and} \qquad Q^{(2)}(0) = 0 \\ \dot Q^{(1)}(0) &= 0 \qquad \text{and} \qquad \dot Q^{(2)}(0) = 1. \end{aligned} \tag{A.8}$$

The amplitude–phase decomposition of $Q^{(1)}(t)$ is taken as

$$\begin{aligned} Q^{(1)} &= \rho\cos(\varphi) \qquad \varphi(0) = 0 \\ \dot Q^{(1)} &= \dot\rho\cos(\varphi) - \Lambda\rho^{-1}\sin(\varphi) \end{aligned} \tag{A.9}$$

with the use of (A.4) *and* (A.7). *The initial condition on* φ *automatically implies the* following companion conditions for the amplitude function:

$$\rho(0) = 1 \qquad \dot\rho(0) = 0. \tag{A.10}$$

The remaining condition on $\dot{\varphi}$ (or Λ) is at our disposal and the purpose is to find a formal condition for specifying it. The amplitude–phase decomposition of $Q^{(2)}(t)$ now has to be taken as

$$\begin{aligned} Q^{(2)} &= \Lambda^{-1}\rho\sin(\varphi) \\ \dot{Q}^{(2)} &= \Lambda^{-1}\dot{\rho}\sin(\varphi) + \rho^{-1}\cos(\varphi) \end{aligned} \tag{A.11}$$

with the same amplitude and phase functions. Let us form the principal fundamental solution matrix $\boldsymbol{Q}$ by

$$\boldsymbol{Q}(t) = \begin{pmatrix} Q^{(1)}(t) & Q^{(2)}(t) \\ \dot{Q}^{(1)}(t) & \dot{Q}^{(2)}(t) \end{pmatrix}. \tag{A.12}$$

It is easily verified, from equations (A.9) and (A.11), that

$$\det \boldsymbol{Q}(t) = 1 \tag{A.13}$$

for all times.

The Floquet theorem [7] for equations of the type (3.1) says that the fundamental solution matrix is subject to a linear period-T_p mapping, $\boldsymbol{M}$, which we define from the initial time propagation as

$$\boldsymbol{Q}(T_p) = \boldsymbol{Q}(0)\boldsymbol{M}. \tag{A.14}$$

The uniqueness of this mapping implies that at any time $t = nT_p$, the solution matrix is given by:

$$\boldsymbol{Q}(nT_p) = \boldsymbol{Q}(0)\boldsymbol{M}^n \tag{A.15}$$

where for negative n the mapping is composed by the inverse $\boldsymbol{M}^{-1}$. From the explicit solutions and the initial conditions we find an expression for the map in terms of the amplitude and phase functions:

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{Q}(T_p) \\ &= \begin{pmatrix} \rho(T_p)\cos\varphi(T_p) & \Lambda^{-1}\rho(T_p)\sin\varphi(T_p) \\ \dot{\rho}(T_p)\cos\varphi(T_p) - \Lambda\rho^{-1}(T_p)\sin\varphi(T_p) & \Lambda^{-1}\dot{\rho}(T_p)\sin\varphi(T_p) + \rho^{-1}(T_p)\cos\varphi(T_p) \end{pmatrix} \end{aligned} \tag{A.16}$$

and

$$\begin{aligned} \boldsymbol{M}^{-1} &= \boldsymbol{Q}(-T_p) \\ &= \begin{pmatrix} \rho(T_p)\cos\varphi(T_p) & -\Lambda^{-1}\rho(T_p)\sin\varphi(T_p) \\ -\dot{\rho}(T_p)\cos\varphi(T_p) + \Lambda\rho^{-1}(T_p)\sin\varphi(T_p) & \Lambda^{-1}\dot{\rho}(T_p)\sin\varphi(T_p) + \rho^{-1}(T_p)\cos\varphi(T_p) \end{pmatrix} \end{aligned} \tag{A.17}$$

with the property

$$\det \boldsymbol{M} = 1. \tag{A.18}$$

In order to obtain the expression for the inverse in (A.17) we have used the fact that $\rho(-t) = \rho(t)$, $\dot{\rho}(-t) = \dot{\rho}(t)$ and $\varphi(-t) = -\varphi(t)$ stated in appendix 1.

The consistency of the expressions (A.16) and (A.17) has to be analysed next. The formal inverse of (A.16) is given by

$$M^{-1} = [Q(T_p)]^{-1} = \begin{pmatrix} \Lambda^{-1}\dot{\rho}(T_p)\sin\varphi(T_p) + \rho^{-1}(T_p\cos\varphi(T_p) & -\Lambda^{-1}\rho(T_p)\sin\varphi(T_p) \\ -\dot{\rho}(T_p)\cos\varphi(T_p) + \Lambda\rho^{-1}(T_p)\sin\varphi(T_p) & \rho(T_p)\cos\varphi(T_p) \end{pmatrix} \tag{A.19}$$

which, so far, differs in form from formula (A.17). Numerically, however, the matrix elements of (A.17) and (A.19) are the same, since the representation is exact for all parameters Λ. To make them *formally* identical we require

$$\rho^2(T_p) = 1 \qquad \text{and} \qquad \dot{\rho}(T_p) = 0. \tag{A.20}$$

In other words, $\rho(t)$ has to be a *periodic solution* of the Milne equation. This requirement forces the angular momentum parameter to assume a specific value $\Lambda = \Lambda_p$.

The narrow-tube mapping matrix can now be simplified and further decomposed in the following way:

$$M = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda_p \end{pmatrix} \begin{pmatrix} \cos(WT_p) & \sin(WT_p) \\ -\sin(WT_p) & \cos(WT_p) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Lambda_p^{-1} \end{pmatrix} \tag{A.21}$$

where

$$W = \frac{1}{T_p}\int_0^{T_p} \dot{\varphi}\,\mathrm{d}t = \frac{\Lambda_p}{T_p}\int_0^{T_p} \rho^{-2}(t)\,\mathrm{d}t \tag{A.22}$$

is the angular winding frequency of the map. The period-T_p map of phase-space points is seen to be a simple similarity transformation of a pure rotation (in the negative sense). Once the map has been established it can be applied repeatedly to any point (Q_0, P_0) of phase space represented by a column vector according to:

$$\begin{pmatrix} Q_n \\ P_n \end{pmatrix} = M^n \begin{pmatrix} Q_0 \\ P_0 \end{pmatrix} \tag{A.23}$$

where

$$M^n = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda_p \end{pmatrix} \begin{pmatrix} \cos(nWT_p) & \sin(nWT_p) \\ -\sin(nWT_p) & \cos(nWT_p) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \Lambda_p^{-1} \end{pmatrix} \tag{A.24}$$

hence, generating a cross-section contour of a particular vortex tube.

References

[1] Tabor M 1989 *Chaos and Integrability in Non-linear Dynamics* (New York: Wiley)
[2] Breuer H P and Holthaus M 1991 *Ann. Phys.* **211** 249
[3] Bensch F 1993 Semiclassical quantization and decay dynamics in time-periodic systems *Doctoral Thesis* Kaiserslautern
[4] Bensch F, Korsch H J, Mirbach B and Ben-Tal N 1992 *J. Phys. A: Math. Gen.* **25** 6761
[5] Thylwe K-E and Bensch F 1994 *J. Phys. A: Math. Gen.* **27** 5673
[6] Miller W H 1975 *J. Chem. Phys.* **63** 996
[7] Arnold V I 1984 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[8] Lewis H R 1968 *J. Math. Phys.* **9** 1976
[9] Kaushal R S and Korsch H J 1981 *J. Math. Phys.* **22** 1904